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Closure Approximation Error in the Mean Solution of Stochastic Differential Equations by the Hierarchy Method

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The iterative or stochastic Green's function method of solution of stochastic differential equations is used to find the error terms in the solution and mean solution due to truncation in the hierarchy method. A comparison is made of solutions by the iterative and the hierarchy method.

KEY WORDS: Stochastic differential equation; hierarchy method; closure approximation; truncation error; iterative method; stochastic Green's function.

Consider the nth-order linear stochastic differential equation[1,2] $\mathscr{L}y = x$, where $\mathscr{L}$ is an nth-order stochastic (differential) operator given by $\mathscr{L} = \sum_{\nu=0}^{n} a_\nu(t, \omega)\, d^\nu/dt^\nu$, where the $a_\nu(t, \omega)$ $[t \in T, \omega \in (\Omega, \mathscr{F}, \mu)]$, a probability space, may be stochastic processes for $\nu = 0, 1,..., n - 1$; $a_n > 0$; and $x(t, \omega)$ is a stochastic process on $T \times \Omega$. Let $\mathscr{L} = L + R$ with L a deterministic (and invertible) operator given by $L = \sum_{\nu=0}^{n} \langle a_\nu(t, \omega)\rangle\, d^\nu/dt^\nu$ and $R = \sum_{\nu=0}^{n-1} \alpha_\nu(t, \omega)\, d^\nu/dt^\nu$, where $a_\nu = \langle a_\nu\rangle + \alpha_\nu$. Assume $\langle R\rangle = 0$.

The commonly used hierarchy method[2] must resort to a closure approximation to truncate the hierarchy in order to obtain a solution. This is known to lead to erroneous results,[3] but the precise error in the solution or the mean solution has not been evaluated. To see how this error arises we rewrite the equation in the form

$$y = L^{-1}x - L^{-1}Ry$$

and operate from the left by R. Then

$$Ry = RL^{-1}x - RL^{-1}Ry$$

Averaging the equation for y, we have

$$\langle y\rangle = L^{-1}\langle x\rangle - L^{-1}\langle Ry\rangle = L^{-1}\langle x\rangle - L^{-1}\langle RL^{-1}Ry\rangle \qquad (1)$$

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by substituting for $\langle Ry\rangle$ from the average of the second equation. If the usual closure approximation is made at this first stage, we have

$$\langle RL^{-1}Ry\rangle = \langle RL^{-1}R\rangle\langle y\rangle$$

which is clearly equivalent to neglecting the fluctuations of $RL^{-1}R$. One wishes to know what error in y results.

Closure approximations at the second stage means setting

$$\langle RL^{-1}RL^{-1}Ry\rangle = \langle RL^{-1}RL^{-1}R\rangle\langle y\rangle$$

neglecting the fluctuations of $RL^{-1}RL^{-1}R$, etc., for higher stage closures.[3]

The iterative method of Adomian is in many ways more satisfactory for a solution since it involves no closure approximations. Further, computer results from the dissertation of Elrod[4] excerpted here show excellent validity for the iterative method. It is natural therefore to contemplate its use to provide an answer to the question of error involved in closure approximation by the hierarchy method. Of course, one may object that one approximate method is being used to evaluate another. However, the procedure is reasonable because there is no measure of the error in the solution by hierarchy, the iterative method is accurate, and the procedure here provides theoretical insight into closure approximation errors.

The solution by the iterative method has been given[2] as $y = \sum_{i=0}^{\infty} (-1)^i y_i$, where

$$\begin{aligned}
y_0 &= L^{-1}x \\
y_1 &= L^{-1}RL^{-1}x \\
y_2 &= L^{-1}RL^{-1}RL^{-1}x = (L^{-1}R)^2L^{-1}x \\
y_3 &= L^{-1}RL^{-1}RL^{-1}RL^{-1}x = (L^{-1}R)^3L^{-1}x
\end{aligned}$$

etc., or $y_i = (L^{-1}R)^iL^{-1}x$.

Let us therefore examine the left and right sides of the last equation using the iterative solution:

$$\begin{aligned}
y &= \sum (-1)^{n-1}(L^{-1}R)^{n-1}L^{-1}x \\
\langle RL^{-1}Ry\rangle &= \langle RL^{-1}R[L^{-1}x - L^{-1}RL^{-1}x + L^{-1}RL^{-1}RL^{-1}x \\
&\quad - L^{-1}RL^{-1}RL^{-1}RL^{-1}x \cdots]\rangle \\
&= \langle RL^{-1}R\rangle L^{-1}\langle x\rangle - \langle RL^{-1}RL^{-1}R\rangle L^{-1}\langle x\rangle \\
&\quad + \langle RL^{-1}RL^{-1}RL^{-1}R\rangle L^{-1}\langle x\rangle \\
&\quad - \langle RL^{-1}RL^{-1}RL^{-1}RL^{-1}R\rangle L^{-1}\langle x\rangle + \cdots \\
\langle RL^{-1}R\rangle\langle y\rangle &= \langle RL^{-1}R\rangle[L^{-1}\langle x\rangle - L^{-1}\langle R\rangle L^{-1}\langle x\rangle \\
&\quad + L^{-1}\langle RL^{-1}R\rangle L^{-1}\langle x\rangle - L^{-1}\langle RL^{-1}RL^{-1}R\rangle L^{-1}\langle x\rangle \cdots] \\
&= \langle RL^{-1}R\rangle L^{-1}\langle x\rangle + \langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}R\rangle L^{-1}\langle x\rangle \\
&\quad - \langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}RL^{-1}R\rangle L^{-1}\langle x\rangle \cdots
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\langle RL^{-1}Ry\rangle - \langle RL^{-1}R\rangle\langle y\rangle \\
&\quad = \left[\left\langle RL^{-1}R\sum(-1)^{n-1}(L^{-1}R)^{n-1}\right\rangle \right. \\
&\qquad \left. - \langle RL^{-1}R\rangle \sum(-1)^{n-1}\langle(L^{-1}R)^{n-1}\rangle\right]L^{-1}\langle x\rangle \\
&\quad = \{[-\langle RL^{-1}RL^{-1}R\rangle + \langle RL^{-1}RL^{-1}RL^{-1}R\rangle \\
&\qquad - \langle RL^{-1}RL^{-1}RL^{-1}RL^{-1}R\rangle \cdots] \\
&\qquad - [\langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}R\rangle \\
&\qquad + \langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}RL^{-1}R\rangle \cdots]\}L^{-1}\langle x\rangle
\end{aligned} \tag{2}$$

Thus the truncation at this stage results in the above error (within the neglected terms). If a Gaussian assumption is made to eliminate the odd terms, then within the neglected terms we have

$$\begin{aligned}
&\langle RL^{-1}Ry\rangle - \langle RL^{-1}R\rangle\langle y\rangle \\
&\quad = (\langle RL^{-1}RL^{-1}RL^{-1}R\rangle - \langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}R\rangle + \cdots)L^{-1}\langle x\rangle
\end{aligned} \tag{3}$$

Of course if $RL^{-1}R = \langle RL^{-1}R\rangle$ the approximation is valid. If not, the error in y in making the truncation at this stage is given by (3). What then is the error in $\langle y\rangle$ which results? We have from (1)

$$\langle y\rangle = L^{-1}\langle x\rangle - L^{-1}\langle RL^{-1}Ry\rangle$$

Then substituting for $\langle RL^{-1}Ry\rangle$ not just $\langle RL^{-1}R\rangle\langle y\rangle$, but the neglected error as well from (3), we have

$$\begin{aligned}
&\langle y\rangle = L^{-1}\langle x\rangle - L^{-1}\langle RL^{-1}R\rangle\langle y\rangle \\
&\quad - (\langle RL^{-1}RL^{-1}RL^{-1}R\rangle - \langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}R\rangle + \cdots)L^{-1}\langle x\rangle
\end{aligned} \tag{4}$$

Hence

$$\begin{aligned}
&[1 + L^{-1}\langle RL^{-1}R\rangle]\langle y\rangle \\
&\quad = [1 - (\langle RL^{-1}RL^{-1}RL^{-1}R\rangle - \langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}R\rangle + \cdots)]L^{-1}\langle x\rangle
\end{aligned} \tag{5}$$

is an equation for $\langle y\rangle$. It is of course possible to similarly find the error in $\langle y\rangle$ due to truncation at higher stages of the hierarchy or to find the error in the correlation or covariance of y due to closure approximations at various levels.

Evaluation of such expressions depends on the form of R, the Green's function $l(t, \tau)$ for the operator L, and the type of process involved. Thus, to evaluate the expression [from (5)]

$$\langle RL^{-1}RL^{-1}RL^{-1}R\rangle - \langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}R\rangle \tag{6}$$

suppose $R = \alpha(t, \omega)$, which we write $\alpha(t)$. Then

$$L^{-1}R = \int_0^t l(t, \tau)\alpha(\tau)\, d\tau$$

and proceeding in a straightforward manner we obtain

$$RL^{-1}R = \int_0^t l(t, \tau)\alpha(t)\alpha(\tau)\, d\tau$$

$$L^{-1}RL^{-1}R = \int_0^t \int_0^\tau l(t, \tau)l(\tau, \gamma)\alpha(\tau)\alpha(\gamma)\, d\gamma\, d\tau$$

$$RL^{-1}RL^{-1}R = \int_0^t \int_0^\tau l(t, \tau)l(\tau, \gamma)\alpha(t)\alpha(\tau)\alpha(\gamma)\, d\gamma\, d\tau$$

$$L^{-1}RL^{-1}RL^{-1}R = \int_0^t \int_0^\tau \int_0^\gamma l(t, \tau)l(\tau, \gamma)l(\gamma, \sigma) \times \alpha(\tau)\alpha(\gamma)\alpha(\sigma)\, d\sigma\, d\gamma\, d\tau$$

$$RL^{-1}RL^{-1}RL^{-1}R = \int_0^t \int_0^\tau \int_0^\gamma l(t, \tau)l(\tau, \gamma)l(\gamma, \sigma) \times \alpha(t)\alpha(\tau)\alpha(\gamma)\alpha(\sigma)\, d\sigma\, d\gamma\, d\tau$$

Hence

$$\langle RL^{-1}RL^{-1}RL^{-1}R\rangle = \int_0^t \int_0^\tau \int_0^\gamma l(t, \tau)l(\tau, \gamma)l(\gamma, \sigma) \times \langle\alpha(t)\alpha(\tau)\alpha(\gamma)\alpha(\sigma)\rangle\, d\sigma\, d\gamma\, d\tau \tag{7}$$

and

$$\langle RL^{-1}R\rangle = \int_0^t l(t, \tau)\langle\alpha(t)\alpha(\tau)\rangle\, d\tau$$

$$L^{-1}\langle RL^{-1}R\rangle = \int_0^t \int_0^\tau l(t, \tau)l(\tau, \gamma)\langle\alpha(\tau)\alpha(\gamma)\rangle\, d\gamma\, d\tau$$

$$\langle RL^{-1}R\rangle L^{-1}\langle RL^{-1}R\rangle = \int_0^t l(t, \tau)\langle\alpha(t)\alpha(\tau)\rangle\, d\tau \times \int_0^t \int_0^\tau l(t, \tau)l(\tau, \gamma)\langle\alpha(\tau)\alpha(\gamma)\rangle\, d\gamma\, d\tau \tag{8}$$

Assuming Gaussian behavior for α, the ensemble average in (7) can be written in terms of the correlation of α, i.e., R_α; hence (7) and (8) and therefore (6) can be evaluated by assuming a particular correlation function, e.g., for white noise or an Ornstein–Uhlenbeck process. Thus in a particular stochastic differential equation with particular stochastic behavior, the error in making closure approximation at various stages could be found.

It is preferable to avoid the use of hierarchy methods altogether and use the iterative or sgf procedure. If one persists in its use, the above errors should be calculated to see if it is reasonable to use the method.

Interpretation. A convenient physical interpretation exists which provides insight into a comparison of the hierarchy and iterative methods and into the error in the mean solution due to the closure approximation in the hierarchy equations. Consider $\mathscr{L}y = x$ or the equivalent equation (when $\mathscr{L}^{-1} = \mathscr{H}$ exists) $y = \mathscr{H}x$. The iterative procedure gives us $\mathscr{H}$ and finally

$$y = \sum (-1)^{n-1}(L^{-1}R)^{n-1}L^{-1}x$$

This can be viewed as a matrix product of an infinite row matrix and an infinite column matrix,

$$y = \begin{vmatrix} L^{-1} \\ (-1)(L^{-1}R)L^{-1} \\ (-1)^2(L^{-1}R)^2L^{-1} \\ \vdots \\ (-1)^{n-1}(L^{-1}R)^{n-1}L^{-1} \\ \vdots \end{vmatrix} \cdot |x, x, \ldots|$$

For any approximate solution for y, say ϕ_n, we have the finite matrix product

$$\varphi_n = \begin{vmatrix} L^{-1} \\ (-1)(L^{-1}R)L^{-1} \\ (-1)^2(L^{-1}R)^2L^{-1} \\ \vdots \\ (-1)^{n-1}(L^{-1}R)L^{-1} \end{vmatrix} \cdot |x, \ldots, x|$$

where the column matrix is the n-tuple.

The equation $\mathscr{L}y = x$ has been decomposed previously into $(L + R)y = x$, or under suitable assumptions

$$\begin{aligned} y &= L^{-1}x - L^{-1}Ry \\ &= L^{-1}x - L^{-1}R(L^{-1}x - L^{-1}R\{L^{-1}x \\ &\quad - L^{-1}R[L^{-1}x \cdots L^{-1}R(L^{-1}x - L^{-1}Ry)]\} \cdots) \\ &= L^{-1}x - (L^{-1}R)(L^{-1}x) + (L^{-1}R)^2L^{-1}x \\ &\quad - (L^{-1}R)^3L^{-1}x + \cdots + (-1)^{n-1}(L^{-1}R)^{n-1}y + \cdots \end{aligned}$$

Writing this also as a matrix product of an infinite row and infinite column matrices, we have

$$y = \begin{vmatrix} L^{-1} \\ (-1)(L^{-1}R)L^{-1} \\ (-1)^2(L^{-1}R)^2L^{-1} \\ \vdots \\ (-1)^{n-1}(L^{-1}R)L^{-1} \\ \vdots \end{vmatrix} \cdot (x, x, x, \ldots, y, \ldots)$$

For an approximate solution again we have

$$\psi_n = \begin{vmatrix} L^{-1} \\ (-1)(L^{-1}R)L^{-1} \\ (-1)^2(L^{-1}R)L^{-1} \\ \vdots \\ (-1)^{n-1}(L^{-1}R)^{n-1}L^{-1} \end{vmatrix} \cdot \underset{n \text{ tuple}}{(x, x, x, \ldots, x, y)}$$

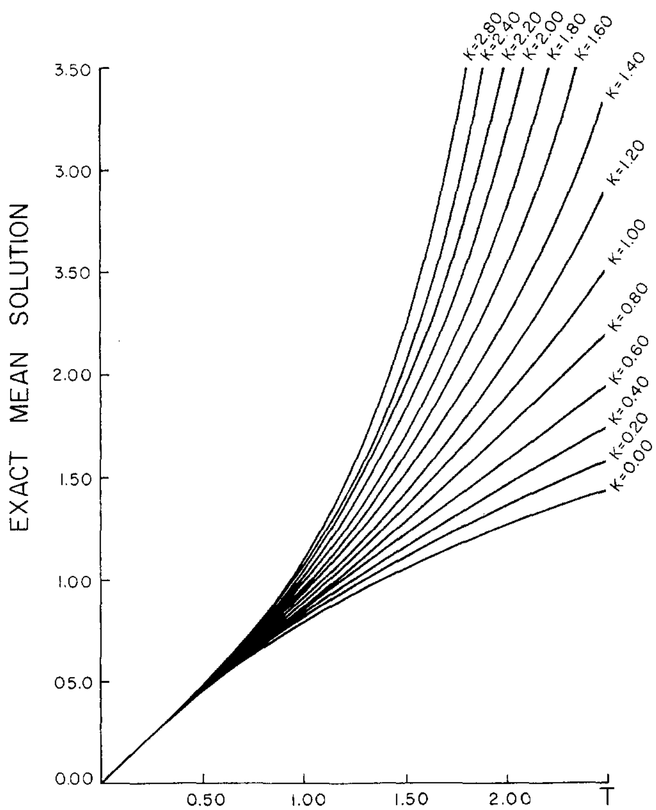

Fig. 1. Analytic solution for $\langle y(t) \rangle$.

From the matrix product representation of the solution process, one can interpret the solution as obtained from a complex system of many input/output systems.

In Adomian's iterative expansion, we view the many inputs as all equal to x and outputs with increasing complexity given by the column matrix.

In the closure approximation of the hierarchy method, the input is nearly identical to that of the iterative method, except for the last entry in the row matrix. This is where the error arises. If the hierarchy was carried out indefinitely, the two methods would yield the same result. The fluctuation in the input would be at infinity. However, in any finite system, i.e., with closure, the fluctuation in the input will be at the last entry of the row matrix, which affects the approximate solution appreciably. Elrod's dissertation compared results for iterative and hierarchy methods with an exact solution for a first-order linear stochastic differential equation. (This work is continuing at CAM and will be published.) The equation considered is

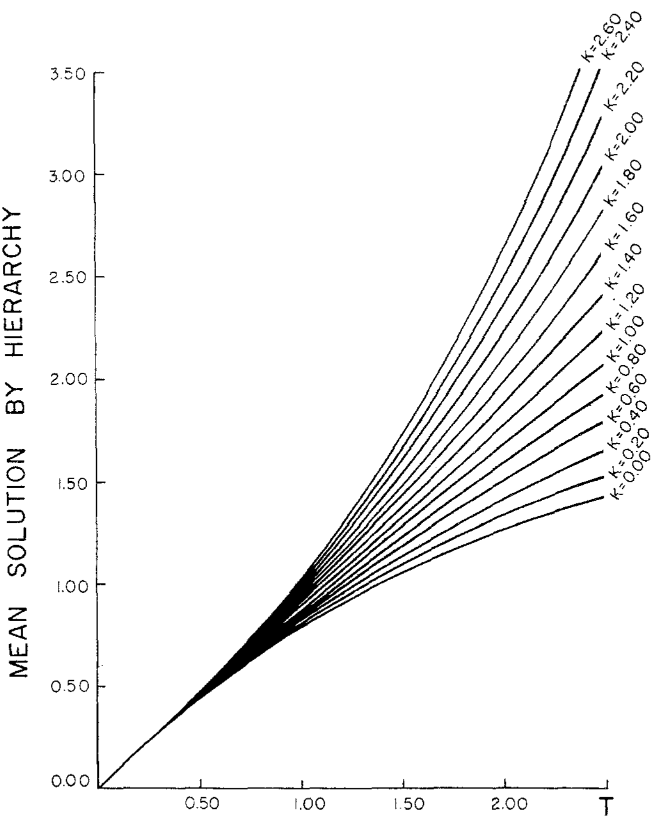

Fig. 2. Hierarchy solution for $\langle y(t)\rangle$.

Table I. Comparison of Solutions[a]

T \ K		0.000	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000
0.00	E	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	P	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	A	0.000	−0.000	−0.000	−0.000	−0.000	−0.000	−0.000	0.000	0.000	0.000	0.000
0.50	E	0.442	0.445	0.449	0.452	0.455	0.458	0.461	0.465	0.468	0.472	0.475
	P	0.442	0.445	0.448	0.451	0.454	0.457	0.460	0.463	0.466	0.469	0.472
	A	0.442	0.445	0.449	0.452	0.455	0.458	0.461	0.465	0.468	0.472	0.475
1.00	E	0.787	0.806	0.825	0.846	0.867	0.890	0.913	0.938	0.963	0.990	1.018
	P	0.787	0.804	0.821	0.838	0.856	0.837	0.892	0.910	0.929	0.948	0.967
	A	0.787	0.806	0.825	0.846	0.867	0.890	0.913	0.938	0.963	0.990	1.018
1.50	E	1.055	0.104	1.158	1.216	1.279	1.349	1.425	1.509	1.600	1.701	1.813
	P	1.055	1.097	1.140	1.184	1.230	1.278	1.327	1.378	1.431	1.485	1.541
	A	1.055	1.104	1.158	1.216	1.279	1.349	1.425	1.509	1.600	1.701	1.812
2.00	E	1.264	1.355	1.459	1.579	1.716	1.875	2.059	2.274	2.523	2.815	3.158
	P	1.264	1.338	1.417	1.500	1.587	1.679	1.777	1.879	1.988	2.101	2.221
	A	1.264	1.355	1.459	1.579	1.716	1.875	2.059	2.273	2.522	2.813	3.153
2.50	E	1.427	1.568	1.739	1.945	2.197	2.507	2.889	3.364	3.958	4.705	5.648
	P	1.427	1.538	1.658	1.788	1.928	2.080	2.243	2.419	2.608	2.812	3.031
	A	1.427	1.568	1.739	1.945	2.197	2.506	2.887	3.360	3.946	4.675	5.582
3.00	E	1.554	1.750	2.000	2.321	2.737	3.284	4.009	4.979	6.288	8.068	10.506
	P	1.554	1.703	1.869	2.053	2.256	2.480	2.727	3.000	3.301	3.632	3.996
	A	1.554	1.750	2.000	2.321	2.737	3.281	3.999	4.947	6.202	7.859	10.040

[a] Three solutions for $\langle y(t)\rangle$ as a function of t and k: The exact solution (E); the solution by the perturbation or hierarchy methods with second-order closure (P); and the solution by 12 terms of Adomian's iterative series (A).

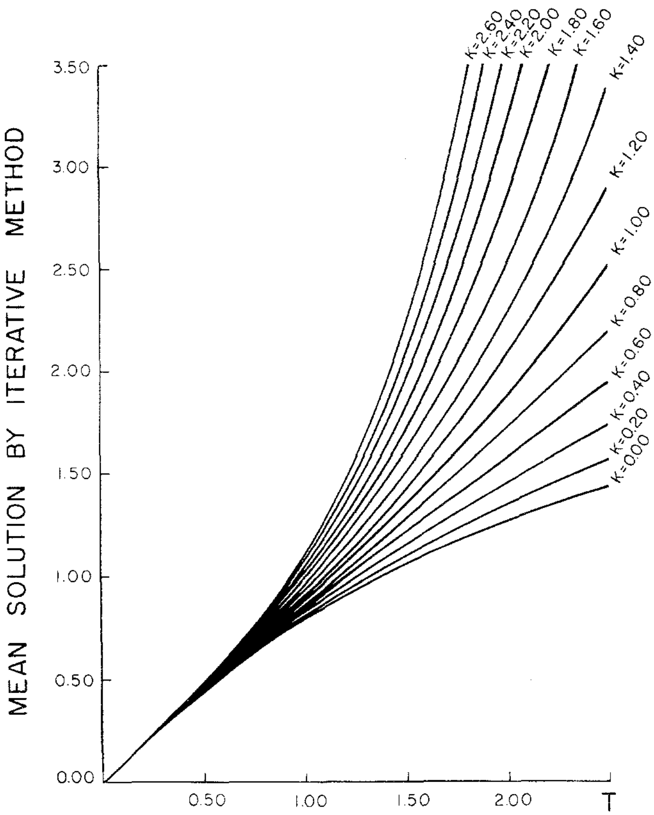

Fig. 3. Twelve terms of Adomian's iterative series solution for $\langle y(t) \rangle$.

$y' + \xi(t)y = \eta(t)$. We assume $\eta(t)$ to be a Gaussian process and $\xi(t)$ to be an Ornstein–Uhlenbeck process with covariance function given by $\text{cov}(t, t') = K \exp(-\alpha|t - t'|)$. Results are shown in Table I and Figs. 1–3.

These results clearly show the superiority of the iterative method and justify its use to evaluate the error in y due to closure.

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